New properties of prolongations of Linear connections on Weil bundles

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Abstract

Let M be a paracompact smooth manifold, A a Weil algebra and M^A the associated Weil bundle. If ∇ is a linear connection on M, we give equivalent definition and the properties of the prolongation ∇^A to M^A equivalent to the prolongation defined by Morimoto. When (M, g) is a pseudo-riemannian manifold, we show that the symmetric tensor g^A of type $(0, 2)$ defined by Okassa is nondegenerated. At the end, we show that , if ∇ is a Levi-Civita connection on (M, g), then ∇^A is torsion-free and g^A is parallel with respect to ∇^A.

1 Introduction

We recall that, in what follows we denote A, a local algebra (in the sense of André Weil) or simply Weil algebra, M a smooth manifold, $C^\infty(M)$ algebra of smooth functions on M and M^A the manifold of infinitely near points of kind A [10]. The triplet (M^A, π, M) is a bundle called bundle of infinitely near points or simply Weil bundle.
If $f : M \longrightarrow \mathbb{R}$ is a smooth function then the application

$$f^A : M^A \longrightarrow A, \xi \longmapsto \xi(f)$$

is also a smooth function . The set, $C^\infty(M^A, A)$ of smooth functions on M^A with values on A, is a commutative algebra over A with unit and the application

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A$$

is an injective homomorphism of algebras. Then, we have:

$$(f + g)^A = f^A + g^A;\ (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A.$$

The map

$$C^\infty(M^A) \times A \longrightarrow C^\infty(M^A, A), (F, a) \longmapsto F \cdot a : \xi \longmapsto F(\xi) \cdot a$$

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is bilinear and induces one and only one linear map

$$\sigma : C^{\infty}(M^A) \otimes A \longrightarrow C^{\infty}(M^A, A).$$

When $(a_\alpha)_{\alpha=1,2,\ldots,\dim A}$ is a basis of A and when $(a^*_\alpha)_{\alpha=1,2,\ldots,\dim A}$ is a dual basis of the basis $(a_\alpha)_{\alpha=1,2,\ldots,\dim A}$, the application

$$\sigma^{-1} : C^{\infty}(M^A, A) \longrightarrow A \otimes C^{\infty}(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a^*_\alpha \circ \varphi)$$

is an isomorphism of A-algebras. That isomorphism does not depend of a choisen basis and the application

$$\gamma : C^{\infty}(M) \longrightarrow A \otimes C^{\infty}(M^A), f \longmapsto \sigma^{-1}(f^A),$$

is a homomorphism of algebras.
If (U, φ) is a local chart of M with coordinate system $(x_1, \ldots, x_n)$, the map

$$\varphi^A : U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), \ldots, \xi(x_n))$$

is a bijection from U^A onto an open set of A^n. In addition, if $(U_i, \varphi_i)_{i\in I}$ is an atlas of M^A, then $(U_i^A, \varphi_i^A)_{i\in I}$ is also an atlas of M^A [2].

1.1 Vector fields on M^A

In [6], we gave another characterization of a vector field on M^A through the above theorem and we also give a writing of a vector field on M^A, in coordinate neighborhood system.

Thus,

Theorem 1 *The following assertions are equivalent:*

1. *A vector field on M^A is a differentiable section of the tangent bundle (TM^A, π_{M^A}, M^A).*
2. *A vector field on M^A is a derivation of $C^{\infty}(M^A)$.*
3. *A vector field on M^A is a derivation of $C^{\infty}(M^A, A)$ which is A-linear.*
4. *A vector field on M^A is a linear map $X : C^{\infty}(M) \longrightarrow C^{\infty}(M^A, A)$ such that*

$$X(f \cdot g) = X(f) \cdot g^A + f^A \cdot X(g), \quad \textit{for any}\, f, g \in C^{\infty}(M).$$

We verify that the $C^{\infty}(M^A, A)$-module $\mathfrak{X}(M^A)$ of vecvector field on M^A is a Lie algebra over A.

Theorem 2 *The map*

$$\mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X, Y) \longmapsto [X, Y] = X \circ Y - Y \circ X$$

is skew-symmetric A-bilinear and defines a structure of A-Lie algebra over $\mathfrak{X}(M^A)$.

In the following, we look at a vector field as a A-linear maps

$$X : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

such that

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi + \varphi \cdot X(\psi), \quad \text{for any } \varphi, \psi \in C^\infty(M^A, A)$$

that is to say

$$\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)].$$

1.2 Prolongations to M^A of vector fields on M.

Proposition 3 *If $\theta : C^\infty(M) \longrightarrow C^\infty(M)$, is a vector field on M, then there exists one and only one A-linear derivation*

$$\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A),$$

such that $\theta^A(f^A) = [\theta(f)]^A$,for any $f \in C^\infty(M)$. Thus, if $\theta, \theta_1, \theta_2$ are vector fields on M and if $f \in C^\infty(M)$, then we have:

1.

$$(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A; (f \cdot \theta)^A = f^A \cdot \theta^A \, and \; [\theta_1, \theta_2]^A = \left[\theta_1^A, \theta_2^A\right].$$

2 Prolongation of linear connections on Weil bundles

In this section, if ∇ [3] is a linear connection on M, we give equivalent definition and the properties of the prolongation ∇^A to M^A equivalent to the prolongation $\overline{\nabla}$ defined by Morimoto. When (M, g) is a pseudo-riemannian manifold, we show that the symmetric tensor g^A of type $(0,2)$ defined by Okassa is nondegenerated. At the end, we show that , if ∇ is a Levi-Civita connection on (M, g), then ∇^A is torsion-free and g^A is parallel with respect to ∇^A.

According [6], if $X : M^A \longrightarrow TM^A$ is a vector field on M^A and if U is a coordinate neighborhood of M with coordinate neighborhood $(x_1, ..., x_n)$, then there exists some functions $f_i \in C^\infty(U^A, A)$ for $i = 1, ..., n$ such that

$$X_{|U^A} = \sum_{i=1}^{n} f_i \left(\frac{\partial}{\partial x_i^A}\right)^A.$$

When (U, φ) is local chart and $(x_1, ..., x_n)$ his local coordinate system. The map
$$U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), ..., \xi(x_n)),$$
is a diffeomorphism from U^A onto an open set on A^n. As
$$\left(\frac{\partial}{\partial x_i}\right)^A : C^\infty(U^A, A) \longrightarrow C^\infty(U^A, A)$$
is such that $\left(\frac{\partial}{\partial x_i}\right)^A (x_j^A) = \delta_{ij}$,we can denote $\frac{\partial}{\partial x_i^A} = \left(\frac{\partial}{\partial x_i}\right)^A$. If $v \in T_\xi M^A$, we can write
$$v = \sum_{i=1}^{n} v(x_i^A) \frac{\partial}{\partial x_i^A}|_\xi \ .$$
If $X \in \mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]$, we have
$$X_{|U^A} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i^A}.$$
with $f_i \in C^\infty(U^A, A)$ for $i = 1, 2, ..., n$.

2.1 Equivalent definitions of derivation laws in $\mathfrak{X}(M^A)$.

In this subsection, we give the definitions of a derivation law in $\mathfrak{X}(M^A) = Der_\mathbb{R}[C^\infty(M^A)]$ and of a derivation law in $\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]$.
Let R be an algebra over a commutative field $\mathbb{K}$. We recall that, a derivation law in a R-module P is a map
$$D : Der_\mathbb{K}(R) \longrightarrow End_\mathbb{K}(P),$$
such that

1. D is R-linear;
2. For any $d \in Der_\mathbb{K}(R)$, the $\mathbb{K}$-endomorphism $D_d : P \longrightarrow P$ satisfies
$$D_d(r \cdot p) = d(r) \cdot p + r \cdot D_d(p)$$
for any $r \in R$, and any $p \in P$, see [4].

We also recall that, a derivation law in the $C^\infty(M)$-module $\mathfrak{X}(M) = Der_\mathbb{R}[C^\infty(M)$ module of vector fields on M is a map
$$D : \mathfrak{X}(M) = Der_\mathbb{R}[C^\infty(M^A)] \longrightarrow End_\mathbb{R}[\mathfrak{X}(M) = Der_\mathbb{R}[C^\infty(M)]],$$
such that

1. D is $C^\infty(M)$-linear;

2. For any $\theta \in \mathfrak{X}(M)$, the $\mathbb{R}$-endomorphism $D_\theta : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ satisfies

$$D_\theta(f \cdot \mu) = \theta(f) \cdot \mu + f \cdot D_\theta(\mu)$$

for any $f \in C^\infty(M)$, and any $\mu \in \mathfrak{X}(M^A)$.

That derivation law defines a linear connection on M, see [9].
Now, we say:

Definition 4 *A derivation law in $\mathfrak{X}(M^A) = Der_{\mathbb{R}}[C^\infty(M^A)]$ is a map*

$$D : \mathfrak{X}(M^A) = Der_{\mathbb{R}}[C^\infty(M^A)] \longrightarrow End_{\mathbb{R}}\left[\mathfrak{X}(M^A) = Der_{\mathbb{R}}[C^\infty(M^A)]\right],$$

such that

1. *D is $C^\infty(M^A)$-linear;*
2. *For any $X \in \mathfrak{X}(M^A)$, the $\mathbb{R}$-endomorphism $D_X : \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A)$ satisfies*
$$D_X(F \cdot Y) = X(F) \cdot Y + F \cdot D_X(Y)$$
for any $F \in C^\infty(M^A)$, and any $Y \in \mathfrak{X}(M^A)$.

Other definition.

In what follows, we denote $\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]$.
We denote $End_A[\mathfrak{X}(M^A)]$ the set of A-endomorphisms of $\mathfrak{X}(M^A)$ i.e the set of maps from $\mathfrak{X}(M^A)$ into $\mathfrak{X}(M^A)$ which are linear over A.

Proposition 5 *The set $End_A[\mathfrak{X}(M^A)]$ is a $C^\infty(M^A, A)$-module.*

Definition 6 *A derivation law in $\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]$. is a map*

$$D : \mathfrak{X}(M^A) \longrightarrow End_{\mathbb{R}}\left[\mathfrak{X}(M^A)\right],$$

such that:

1. *D is $C^\infty(M^A, A)$-linear;*
2. *For any $X \in \mathfrak{X}(M^A)$, the A-endomorphism $D_X : \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A)$ verifies*
$$D_X(\varphi \cdot Y) = X(\varphi) \cdot Y + \varphi \cdot D_X(Y)$$
for any $\varphi \in C^\infty(M^A)$, and any $Y \in \mathfrak{X}(M^A)$.

2.2 The new statement of the Morimoto's prolongation of a linear connection on M.

Theorem 7 *If ∇ is a linear connection on M, then there exists one and only one linear application*

$$\nabla^A : \mathfrak{X}(M^A) \longrightarrow End_A[\mathfrak{X}(M^A)], X \longmapsto \nabla^A_X$$

such that

$$\nabla^A_{\theta^A}\eta^A = (\nabla_\theta \eta)^A ,$$

for any $\theta, \eta \in \mathfrak{X}(M)$.

Proof. If $X \in \mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)$, then

$$X(f^A) = \sum_{\alpha=1}^{\dim A} X^{'}(a^*_\alpha \circ f^A) \cdot a_\alpha = \sum_{\alpha=1}^{\dim A} X(a^*_\alpha \circ f^A) \cdot a_\alpha$$

with $X^{'} \in \mathfrak{X}(M^A) = Der_A[C^\infty(M^A)]$.
Let

$$\overline{\nabla} : \mathfrak{X}(M^A) = Der_{\mathbb{R}}[C^\infty(M^A)] \longrightarrow End_{\mathbb{R}}\left[\mathfrak{X}(M^A) = Der_{\mathbb{R}}[C^\infty(M^A)]\right]$$

be the Morimoto's prolongation to M^A of the linear connection ∇ on M. We denote

$$\nabla^A : \mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)] \longrightarrow End_A\left[\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]\right]$$

the same derivation law in $\mathfrak{X}(M^A) = Der_A[C^\infty(M^A, A)]$. Thus for any θ, $\eta \in \mathfrak{X}(M)$, we have:

$$\begin{aligned}
\left[\nabla^A_{\theta^A}\eta^A\right](f^A) = \sum_{\alpha=1}^{\dim A}\left[\nabla^A_{\theta^A}\eta^A\right]^{'}(a^*_\alpha \circ f^A)\cdot a_\alpha &= \sum_{\alpha=1}^{\dim A}\left[\nabla^A_{(\theta^A)^{'}}(\eta^A)^{'}\right](a^*_\alpha \circ f^A)\cdot a_\alpha \\
&= \sum_{\alpha=1}^{\dim A}\left[(\nabla_\theta\eta)^A\right]^{'}(a^*_\alpha \circ f^A)\cdot a_\alpha \\
&= \sum_{\alpha=1}^{\dim A}\left[(\nabla_\theta\eta)^A\right](a^*_\alpha \circ f^A)\cdot a_\alpha \\
&= \left[(\nabla_\theta\eta)^A\right](f^A),
\end{aligned}$$

for any $f \in C^\infty(M)$, hence

$$\nabla^A_{\theta^A}\eta^A = (\nabla_\theta\eta)^A.$$

■

2.2.1 Torsion of ∇^A.

When ∇ is a linear connection on M, we denote T_∇ the torsion of ∇.

Proposition 8 *If ∇ is a linear connection on M, then the torsion of ∇^A*

$$T_{\nabla^A} : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X,Y) \longmapsto = \nabla^A_X Y - \nabla^A_Y X - [X,Y],$$

is a skew-symmetric $C^\infty(M^A, A)$-bilinear.

Proof.

1. For all vector fields $X, Y, Z \in \mathfrak{X}(M^A)$, we have:

$$\begin{aligned} T_{\nabla^A}(X+Y,Z) &= \nabla^A_{(X+Y)} Z - \nabla^A_Z(X+Y) - [X+Y,Z] \\ &= \nabla^A_X Z + \nabla^A_Y Z - \nabla^A_Z(X) - \nabla^A_Z(Y) - [X,Z] - [Y,Z] \\ &= \nabla^A_X Z - \nabla^A_Z(X) - [X,Z] + \nabla^A_Y Z - \nabla^A_Z(Y) - [Y,Z] \\ &= T_{\nabla^A}(X,Z) + T_{\nabla^A}(Y,Z). \end{aligned}$$

2. For any vector field $X \in \mathfrak{X}(M^A)$, we have:

$$\begin{aligned} T_{\nabla^A}(X,X) &= \nabla^A_X X - \nabla^A_X X - [X,X] \\ &= 0. \end{aligned}$$

3. For any vector fields $X \in \mathfrak{X}(M^A)$ and for any $\varphi \in C^\infty(M^A, A)$, we have

$$\begin{aligned} T_{\nabla^A}(X, \varphi \cdot Y) &= \nabla^A_X \varphi \cdot Y - \nabla^A_{\varphi \cdot Y}(X) - [X, \varphi \cdot Y] \\ &= X(\varphi) \cdot Y + \varphi \cdot \nabla^A_X Y - \varphi \cdot \nabla^A_Y X - X(\varphi) \cdot Y - \varphi \cdot [Y,X] \\ &= \varphi \cdot \nabla^A_X Y - \varphi \cdot \nabla^A_Y X - \varphi \cdot [Y,X] \\ &= \varphi \cdot \left(\nabla^A_X Y - \nabla^A_Y X - [Y,X] \right) \\ &= \varphi \cdot T_{\nabla^A}(X,Y). \end{aligned}$$

Therefore the torsion T_{∇^A} is skew-symmetric $C^\infty(M^A, A)$-bilinear. ■

Proposition 9 *For any $X, Y \in \mathfrak{X}(M^A)$, and if U is coordonate neighborhood of M, then*

$$T_{\nabla^A_{|U^A}}(X_{|U^A}, Y_{|U^A}) = [T_{\nabla^A}(X,Y)]_{|U^A} .$$

Proposition 10 *If ∇ is a linear connection on M, then*

$$T_{\nabla^A}(\theta^A, \eta^A) = [T_\nabla(\theta, \eta)]^A$$

for any $\theta, \eta \in \mathfrak{X}(M)$.

Proof. For any $\theta, \eta \in \mathfrak{X}(M)$, we have:

$$\begin{aligned}
T_{\nabla^A}(\theta^A, \eta^A) &= \nabla^A_{\theta^A}\eta^A - \nabla^A_{\eta^A}\theta^A - [\theta^A, \eta^A] \\
&= [\nabla_\theta \eta]^A - [\nabla_\eta \theta]^A - [\theta, \eta]^A \\
&= \left(\nabla_\theta \eta - \nabla_\eta \theta - [\theta, \eta]^A\right) \\
&= [T_\nabla(\theta, \eta)]^A.
\end{aligned}$$

■

Corollary 11 *If the linear connection ∇ is torsion-free, then ∇^A is also torsion-free*

Proof. Let X, Y be two vector fields M^A and U a coordinate neighborhood of M^A. Then

$$X_{|U^A} = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i^A}; Y_{|U^A} = \sum_{j=1}^n g_j \frac{\partial}{\partial x_j^A}$$

and, we have:

$$\begin{aligned}
[T_{\nabla^A}(X,Y)]_{|U^A} &= T_{\nabla^A_{|U^A}}(X_{|U^A}, Y_{|U^A}) \\
&= T_{\nabla^A_{|U^A}}\left(\sum_{i=1}^n f_i \frac{\partial}{\partial x_i^A}, \sum_{j=1}^n g_j \frac{\partial}{\partial x_j^A}\right) \\
&= \sum_{ij=1}^n f_i g_j T_{\nabla^A_{|U^A}}\left(\frac{\partial}{\partial x_i^A}, \frac{\partial}{\partial x_j^A}\right) \\
&= \sum_{ij=1}^n f_i g_j T_{\nabla^A_{|U^A}}\left(\left(\frac{\partial}{\partial x_i}\right)^A, \left(\frac{\partial}{\partial x_j}\right)^A\right) \\
&= \sum_{ij=1}^n f_i g_j \left[T_{\nabla_{|U}}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right]^A,
\end{aligned}$$

as ∇ is torsion-free that is to say $T_\nabla = 0$, hence $[T_{\nabla^A}(X,Y)]_{|U^A} = 0$. Consequently

$$T_{\nabla^A} = 0.$$

■

2.3 Prolongation of the Levi-Civita connection.

In this subsection we consider (M, g) a pseudo-riemannian manifold, in what follows we study the prolongation of connections to M^A deduce from the Levi-Civita connection on M.

Proposition 12 *[7] Let* $\mathrm{g} : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$ *be a symmetric tensor of type* $(0,2)$ *on* M*. There exists one and only one symmetric tensor* g^A *of type* $(0,2)$ *on* M^A *with value in* A *such that* $\mathrm{g}^A\left(a \cdot \eta^A, b \cdot \theta^A\right) = ab \cdot [\mathrm{g}(\eta,\theta)]^A$ *for any* $a, b \in A$ *and* $\eta, \theta \in \mathfrak{X}(M)$*.*

Following [**?**], we state:

Proposition 13 *When* (M, g) *a pseudo-riemannian manifold, then there exists one and only one* $C^\infty(M^A, A)$*-nondegenerated symmetric bilineat form*

$$\mathrm{g}^A : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A)$$

such that for any vector fields η *and* θ *on* M*,*

$$\mathrm{g}^A\left(\eta^A, \theta^A\right) = [\mathrm{g}(\eta,\theta)]^A$$

where η^A *and* θ^A*mean prolongations to* M^A *of vector fields* η *and* θ*.*

Proof. It is a matter here to show only the nondegeneracy of g^A, the proof is in the same way as in [**?**]. ■

Therefore g^A is a pseudo-riemannian manifold on M^A and confers to M^A the structure of pseudo-riemannian manifold.

Proposition 14 *For any* $X \in \mathfrak{X}(M^A)$*, the map*

$$\nabla^A_X \mathrm{g}^A : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A)$$

such that

$$\nabla^A_X \mathrm{g}^A(Y,Z) = X\left[\mathrm{g}^A(Y,Z)\right] - \mathrm{g}^A\left(\nabla^A_X(Y), Z\right) - \mathrm{g}^A\left(Y, \nabla^A_X Z\right)$$

for any $Y, Z \in \mathfrak{X}(M^A)$ *is a symmetric* $C^\infty(M^A, A)$*-bilinear form.*

Proof.

1. For any $X, Y \in \mathfrak{X}(M^A)$, we have:

$$\begin{aligned}
\nabla^A_X \mathrm{g}^A(Y,Z) &= X\left[\mathrm{g}^A(Y,Z)\right] - \mathrm{g}^A\left(\nabla^A_X(Y), Z\right) - \mathrm{g}^A\left(Y, \nabla^A_X Z\right) \\
&= X\left[\mathrm{g}^A(Z,Y)\right] - \mathrm{g}^A\left(Z, \nabla^A_X(\varphi \cdot Y)\right) - \mathrm{g}^A\left(\nabla^A_X Z, \varphi \cdot Y\right) \\
&= \nabla^A_X \mathrm{g}^A(Z,Y),
\end{aligned}$$

hence $\nabla^A_X \mathrm{g}^A$ is symmetric.

2. Let Y_1, Y_2 and Z be the vector fields in $\mathfrak{X}(M^A)$, we have:

$$\begin{aligned}
\nabla_X^A \mathrm{g}^A(Y_1+Y_2, Z) &= X\left[g^A(Y_1+Y_2, Z)\right] - \mathrm{g}^A\left(\nabla_X^A(Y_1+Y_2), Z\right) - \mathrm{g}^A\left(Y_1+Y_2, \nabla_X^A Z\right) \\
&= X\left[\mathrm{g}^A(Y_1, Z) + g^A(Y_2, Z)\right] - \mathrm{g}^A\left(\nabla_X^A Y_1 + \nabla_X^A Y_2, Z\right) - \mathrm{g}^A\left(Y_1, \nabla_X^A Z\right) \\
&- \mathrm{g}^A\left(Y_2, \nabla_X^A Z\right) \\
&= X\left[g^A(Y_1, Z)\right] + X\left[g^A(Y_2, Z)\right] - g^A\left(\nabla_X^A Y_1, Z\right) - \mathrm{g}^A\left(\nabla_X^A Y_2, Z\right) \\
&- \mathrm{g}^A\left(Y_1, \nabla_X^A Z\right) - \mathrm{g}^A\left(Y_2, \nabla_X^A Z\right) \\
&= X\left[\mathrm{g}^A(Y_1, Z)\right] - \mathrm{g}^A\left(\nabla_X^A Y_1, Z\right) - \mathrm{g}^A\left(Y_1, \nabla_X^A Z\right) + X\left[\mathrm{g}^A(Y_2, Z)\right] \\
&- \mathrm{g}^A\left(\nabla_X^A Y_2, Z\right) - \mathrm{g}^A\left(Y_2, \nabla_X^A Z\right) \\
&= \nabla_X^A \mathrm{g}^A\left(Y_1, Z\right) + \nabla_X^A \mathrm{g}^A(Y_2, Z).
\end{aligned}$$

3. Let Y and Z the vector fields in $\mathfrak{X}(M^A)$ and $\varphi \in C^\infty(M^A, A)$, we have:

$$\begin{aligned}
\nabla_X^A \mathrm{g}^A(\varphi \cdot Y, Z) &= X\left[\mathrm{g}^A(\varphi \cdot Y, Z)\right] - \mathrm{g}^A\left(\nabla_X^A(\varphi \cdot Y), Z\right) - \mathrm{g}^A\left(\varphi \cdot Y, \nabla_X^A Z\right) \\
&= X(\varphi) \cdot \mathrm{g}^A(Y, Z) + \varphi \cdot X\left[\mathrm{g}^A(Y, Z)\right] - \mathrm{g}^A\left(X(\varphi) \cdot Y, Z\right) + \varphi \cdot \mathrm{g}^A\left(\nabla_X^A Y, Z\right) \\
&- \varphi \cdot \mathrm{g}^A\left(Y, \nabla_X^A Z\right) \\
&= X(\varphi) \cdot \mathrm{g}^A(Y, Z) + \varphi \cdot X\left[\mathrm{g}^A(Y, Z)\right] - X(\varphi) \cdot \mathrm{g}^A(Y, Z) - \varphi \cdot \mathrm{g}^A\left(\nabla_X^A Y, Z\right) \\
&- \varphi \cdot \mathrm{g}^A\left(Y, \nabla_X^A Z\right) \\
&= \varphi \cdot X\left[\mathrm{g}^A(Y, Z)\right] - \varphi \cdot \mathrm{g}^A\left(\nabla_X^A Y, Z\right) - \varphi \cdot \mathrm{g}^A\left(Y, \nabla_X^A Z\right) \\
&= \varphi \cdot \nabla_X^A \mathrm{g}^A(Y, Z).
\end{aligned}$$

Therefore, the map $\nabla_X^A \mathrm{g}^A$ is a symmetric $C^\infty(M^A, A)$-bilinear form. ■

Proposition 15 *If ∇ is a linear connection on the pseudo-riemannian manifold (M, g), then we have:*

$$\nabla_{\theta^A}^A \mathrm{g}^A\left(\mu_1^A, \mu_2^A\right) = \left[\nabla_\theta \mathrm{g}\left(\mu_1, \mu_2\right)\right]^A$$

for any $\theta, \mu_1, \mu_2 \in \mathfrak{X}(M^A)$.

Proof. for any $\theta, \mu_1, \mu_2 \in \mathfrak{X}(M^A)$, we have:

$$\begin{aligned}
\nabla_{\theta^A}^A \mathrm{g}^A(\mu_1^A, \mu_2^A) &= \theta^A\left[\mathrm{g}^A(\mu_1^A, \mu_2^A)\right] - \mathrm{g}^A\left(\nabla_{\theta^A}^A \mu_1^A, \mu_2^A\right) - \mathrm{g}^A\left(\mu_1^A, \nabla_{\theta^A}^A \mu_2^A\right) \\
&= \theta^A\left[(\mathrm{g}(\mu_1, \mu_2))^A\right] - \left[\mathrm{g}\left(\nabla_\theta \mu_1, \mu_2\right)\right]^A - \left[\mathrm{g}\left(\mu_1, \nabla_\theta \mu_2\right)\right]^A \\
&= [\theta(\mathrm{g}(\mu_1, \mu_2))]^A - \left[\mathrm{g}\left(\nabla_\theta \mu_1, \mu_2\right)\right]^A - \left[\mathrm{g}\left(\mu_1, \nabla_\theta \mu_2\right)\right]^A \\
&= [\theta[\mathrm{g}(\mu_1, \mu_2)] - \mathrm{g}(\nabla_\theta \mu_1, \mu_2) - \mathrm{g}(\mu_1, \nabla_\theta \mu_2)]^A \\
&= [\nabla_\theta \mathrm{g}(\mu_1, \mu_2)]^A.
\end{aligned}$$

■

Proposition 16 *For any $X, Y, Z \in \mathfrak{X}(M^A)$, and if U is coordinate neighborhood of M, then*

$$\left[\left(\nabla^A_{|U^A}\right)_{|U^A} \mathrm{g}^A_{|U^A})\right]\left(X_{|U^A}, Y_{|U^A}\right) = \left[\nabla^A_X \mathrm{g}^A(Y, Z)\right]_{|U^A}.$$

Corollary 17 *If ∇ is the Levi-Civita connection on the pseudo-riemannian manifold (M, g), then we have:*

$$\nabla^A_X \mathrm{g}^A = 0$$

for any $X \in \mathfrak{X}(M^A)$.

Proof. Let X, Y, Z be vector fields M^A and U a coordinate neighborhood of M^A. Then

$$X_{|U^A} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i^A};\ Y_{|U^A} = \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j^A};\ Z_{|U^A} = \sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k^A}.$$

Thus, we have:

$$\begin{aligned}
\left[\nabla^A_X \mathrm{g}^A(Y, Z)\right]_{|U^A} &= \left[(\nabla^A_{|U^A})_{X_{|U^A}}) \mathrm{g}^A_{|U^A}\left(Y_{|U^A}, Z_{|U^A}\right)\right] \\
&= \left((\nabla^A_{|U^A})_{\left(\sum_{i=1}^n f_i \frac{\partial}{\partial x_i^A}\right)} \mathrm{g}^A_{|U^A}\right)\left(\sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j^A}, \sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k^A}\right) \\
&= \sum_{ijk=1}^{n} f_i g_j h_k \left((\nabla^A_{|U^A})_{\left(\frac{\partial}{\partial x_i^A}\right)} \mathrm{g}^A_{|U^A}\right)\left(\frac{\partial}{\partial x_j^A}, \frac{\partial}{\partial x_k^A}\right) \\
&= \sum_{ijk=1}^{n} f_i g_j h_k \left((\nabla^A_{|U^A})_{\left(\frac{\partial}{\partial x_i}\right)^A} \mathrm{g}^A_{|U^A}\right)\left(\left(\frac{\partial}{\partial x_j}\right)^A, \left(\frac{\partial}{\partial x_k}\right)^A\right) \\
&= \sum_{ijk=1}^{n} f_i g_j h_k \left(\left((\nabla_{|U})_{\left(\frac{\partial}{\partial x_i}\right)} \mathrm{g}_{|U}\right)\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\right)^A.
\end{aligned}$$

As ∇ is the Levi-Civita connection, then $\nabla_\theta \mathrm{g} = 0$, hence $\left[\nabla^A_X \mathrm{g}^A(Y, Z)\right]_{|U^A} = 0$. It follows that,

$$\nabla^A_X \mathrm{g}^A = 0.$$

■

Theorem 18 *If ∇ is a Levi-Civita connection on a pseudo-riemannian manifold (M, g), then ∇^A verifies the following properties:*

1. $T_{\nabla^A} = 0$*;*

2. $\nabla^A_X \mathrm{g}^A = 0$ *for any* $X \in \mathfrak{X}(M^A)$*.*

Proof. The proof is deduced from the corollary **??** and corollary **??**. ∎

Thus ∇^A is a Levi-Civita connection on the pseudo-riemannian manifold (M^A, g^A).

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References

[1] Bossoto, B.G.R., *Structures de Jacobi sur une variété des points proches*, Math. Vesnik. 62, 2 (2010), 155-167.

[2] Bossoto, B.G.R., Okassa, E., *Champs de vecteurs et formes différentielles sur une variété des points proches*, Archivum mathematicum (BRNO), Tomus44 (2008),159-171.

[3] Helgason, S., *Differential Geometry and symmetric spaces,* New York; Academic Press, 1962.

[4] Koszul, J.L., Ramanan, S., *Lectures On Fibre Bundles and Differential Geometry*, Tata Institute of Fundamental Research, Bombay 1960.

[5] Morimoto, A., *Prolongation of connections to bundles of infinitely near points*, J. Diff. Geom, 11 (1976), 479-498.

[6] bb

[7] Okassa, E., *Relèvements de structures symplectiques et pseudo-riemanniennes à des variétés de points proches,* Nagoya Math. J.115 (1989), 63-71.

[8] Okassa, E., *Prolongement des champs de vecteurs à des variétés des points prohes,* Ann. Fac. Sci. Toulouse Math. VIII (3) (1986-1987), 346-366.

[9] Pham-Ham-Mau-Quan, F., *Introduction à la géométrie des variétés différentiables,* Dunod Paris, 1969.

[10] Weil, A., *Théorie des points proches sur les variétés différentiables*, Colloq. Géom. Diff. Strasbourg (1953), 111-117